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A K-THEORETIC INVARIANT FOR DYNAMICAL SYSTEMS

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ABSTRACT. Let (X, T) be a zero-dimensional dynamical system. We consider the quotient group $G = C(X, Z)/B(X, T)$, where $C(X, Z)$ is the group of continuous integer-valued functions on X and $B(X, T)$ is the subgroup of functions of the form $f - f \circ T$. We show that if (X, T) is topologically transitive, then there is a natural order on G which makes G an ordered group. This order structure gives a new invariant for the classification of dynamical systems. We prove that for each n, the number of fixed points of T^n is an invariant of the ordered group G. Then we show how G can be computed as a direct limit of finite rank ordered groups. This is used to study the conditions under which G is a dimension group. Finally we discuss the relation between G and the K_0-group of the crossed product C^*-algebra associated to the system (X, T).

0. INTRODUCTION

A dynamical system (X, T) consists of a compact metric space X and a homeomorphism $T: X \to X$. Two systems (X, T) and (Y, S) are said to be (topologically) *conjugate* to each other if there exists a homeomorphism $h: X \to Y$ such that $h \circ T = S \circ h$. A central problem in this subject is to find new invariants for distinguishing dynamical systems up to conjugacy. This is particularly evident in the theory of Markov chains [15], where one can find simple examples for which the conjugacy problem is still unsolved. In this paper we study systems for which the spaces are *zero dimensional*. A space X is called zero dimensional if the topology on X has a basis of sets which are both closed and open (*clopen*). Given a zero-dimensional system (X, T), we consider the quotient group $G(X, T) = C(X, Z)/B(X, T)$, where $C(X, Z)$ is the group of continuous functions from X to the integers Z and $B(X, T)$ is the subgroup of functions of the form $f - f \circ T$. When the system (X, T) under study is clear, we will write G for $G(X, T)$. Parry and Tuncel have shown [15] that if (X, T) is a Markov chain, then, except in some trivial cases, G is isomorphic to the free abelian group on countably infinite many generators. Thus, this group cannot generally be used to distinguish Markov chains.

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In §1 we consider an ordering on G defined by $G^+ = \{[f]: f(x) \geq 0$ for all $x \in X\}$. It is shown that if (X, T) is topologically transitive, then we have an ordered group (G, G^+). In §2, we prove that for each $n \geq 1$, the number of fixed points of T^n is an invariant of the ordered group G. This shows that the zeta function of subshifts and topological entropy of Markov chains are invariants of G. Then we consider the conjugacy problems of Markov chains. We show that if A, B are two irreducible matrices such that G_A and G_B are order isomorphic, then (X_A, T_A) and (X_B, T_B) are almost topologically conjugate as defined by Adler and Marcus [1]. In §3, we show how $G(X, T)$ can be computed as a direct limit of finite rank ordered groups. Then we study the conditions under which $G(X, T)$ is a dimension group [10]. We conclude with some remarks on the relation between $G(X, T)$ and the K_0-group [3, 10] of the crossed product [16] $Z \times_T C(X)$, of the system (X, T). We will use [9, 15] for our references on finite shifts and Markov chains and [3, 10 and 16] for K-theory and C^*-algebras.

Most of the results in this paper are done in my Ph.D. thesis at UCLA. I wish to express my deepest gratitude to my advisor, Professor Edward G. Effros, for his guidance, encouragement and criticisms throughout the course of my study.

1. The ordered group $G(X, T)$

Let (X, T) be a zero-dimensional dynamical system. A *partition* of X is a collection $\{O_i: i \in I\}$ of nonempty, pairwise disjoint, clopen subsets of X such that $\bigcup_{i\in I} O_i = X$. Since X is compact, it follows that I is finite. We have

Lemma 1.1. *For each $f \in C(X, Z)$, there exist a partition $\{O_i: 1 \leq i \leq n\}$ of X and a sequence of integers $\{a_i: 1 \leq i \leq n\}$ such that $f = \sum_{i=1}^n a_i \chi_{O_i}$, where χ_O is the characteristic function of O.*

Proof. Since X is compact, $f(X)$ is a finite subset $\{a_i: 1 \leq i \leq n\}$ of Z. For each i, let $O_i = f^{-1}(\{a_i\})$. Then each O_i is clopen and $\{O_i\}$ is a partition of X such that $f = \sum_{i=1}^n a_i \chi_{O_i}$. □

From now on, for $f \in C(X, Z)$, when we write $f = \sum_{i=1}^n a_i \chi_{O_i}$, it will be understood that $a_i \in Z$ and $\{O_i: 1 \leq i \leq n\}$ is a partition of X. $C(X, Z)$ is a group under the usual addition. Let $G = G(X, T)$ be the quotient group $C(X, Z)/(B(X, T)$, where

$$B(X, T) = \{f - f \circ T: f \in C(X, Z)\}.$$

Define

$$G^+ = \{[f] \in G: f(x) \geq 0 \text{ for all } x \in X\}.$$

Proposition 1.2. *Let $f \in C(X, Z)$. Then*

(a) $[f] \in G^+$ *if and only if there exist a partition $\{O_i: i = 1, \dots, n\}$ and integers a_i, t_i, $1 \leq i \leq n$, such that $f = \sum_{i=1}^n a_i \chi_{O_i}$ and $a_i + t_i - t_j \geq 0$ for all i, j with $O_i \cap T^{-1}(O_j) \neq \varnothing$.*

(b) $[f] = 0$ *in* G *if and only if there exist a partition* $\{O_i : i = 1, \dots, n\}$ *and integers* a_i, t_i, $1 \le i \le n$, *such that* $f = \sum_{i=i}^n a_i \chi_{O_i}$ *and* $a_i + t_i - t_j = 0$ *for all* i, j *with* $O_i \cap T^{-1}(O_j) \neq \varnothing$.

Proof. (a) By definition, $f \in C(X, Z)$ satisfies $[f] \in G^+$ if and only if there exists $h \in C(X, Z)$ such that

$$(f + h - h \circ T)(x) \ge 0 \quad \text{for all } x \in X. \tag{1.1}$$

Since X is zero dimensional, there exist a partition $\{O_i : i = 1, \dots, n\}$ and integers a_i, t_i, $1 \le i \le n$, such that

$$f = \sum_{i=1}^n a_i \chi_{O_i} \quad \text{and} \quad h = \sum_{i=1}^n t_i \chi_{O_i}.$$

Then (1.1) holds if and only if

$$\left(\sum_{i=1}^n a_i \chi_{O_i} + t_i \chi_{O_i} - t_i \chi_{O_i} \circ T \right)(x) \ge 0 \quad \text{for all } x \in X$$

$$\Leftrightarrow \left(\sum_{i=1}^n a_i \chi_{O_i} + t_i \chi_{O_i} - t_i \chi_{T^{-1}(O_i)} \right)(x) \ge 0 \quad \text{for all } x \in X.$$

If $x \in O_i \cap T^{-1}(O_j)$ for some i, j, then the left-hand side of the above inequality is equal to $a_i + t_i - t_j$. Since

$$X = \left(\bigcup_{i=1}^n O_i \right) \cap \left(\bigcup_{j=1}^n T^{-1}(O_j) \right) = \bigcup_{1 \le i, j \le n} (O_i \cap T^{-1}(O_j))$$

is proven. Replacing "$\ge$" by "$=$" in the above argument, we get (b). □

The conditions in Proposition 1.2 motivate the following

Definition 1.3. For each partition $\{O_i : i = 1, \dots, n\}$, we define a directed graph [4] on $V = \{1, \dots, n\}$ by letting $i \to j$ if $O_i \cap T^{-1}(O_j) \neq \varnothing$.

Given a directed graph $(V, \to)$ and $i, j \in V$, we write $i \twoheadrightarrow j$ if there is a sequence $s = (i_1, \dots, i_m)$, $m > 1$, such that $i_1 = i$, $i_m = j$ and $i_1 \to i_2 \to \cdots \to i_m$. s is called a *path* from i to j. If there is a path $(i_1, \dots, i_m)$ for which $i_1 = i_m$, then $s = (i_1, \dots, i_{m-1})$ is called a *cycle*. If the directed graph comes from a partition P of X, then we just say s is a path (or cycle) of P. Given a directed graph $(V, \to)$ and $a = (a_i)_{i \in V}$ with $a_i \in Z$, define

$$\sum_s a = \sum_{j=1}^m a_{i_j} \quad \text{for } s = (i_1, \dots, i_m),\ i_j \in V.$$

If $f \in C(X, Z)$ is given by $f = \sum_{i=1}^n a_i \chi_{O_i}$, then we put $a = (a_1, \dots, a_n)$ and let $\sum_s f = \sum_s a$.

Lemma 1.4. *Let $(V, \to)$ be a directed graph and $a = (a_i)_{i\in V}$ such that $\sum_s a = 0$ for every cycle s. Then for all $i, j \in V$ with $j \twoheadrightarrow i$, we have $\sum_{s_1} a = \sum_{s_2} a$ for any two paths s_1, s_2 from i to j.*

Proof. Let $s_1 = (i_1, \ldots, i_m)$ and $s_2 = (j_1, \ldots, j_n)$ be paths from i to j and $s_3 = (k_1, \ldots, k_r)$ a path from j to i. Then $s = (i_1, \ldots, i_m, k_2, \ldots, k_{r-1})$ and $s' = (j_1, \ldots, j_n, k_2, \ldots, k_{r-1})$ are cycles, so we have

$$\sum_{s_1} a + \sum_{t=2}^{r-1} a_{k_t} = \sum_{s} a = 0 = \sum_{s'} a = \sum_{s_2} a + \sum_{t=2}^{r-1} a_{k_t} \Rightarrow \sum_{s_1} a = \sum_{s_2} a. \qquad \square$$

Definition 1.5. A system (X, T) is said to be *topologically transitive* if there exists $x \in X$ such that $\{T^n(x) : n \in Z\}$ is dense in X. Equivalently [25], if for any two nonempty open sets U, V, there exists $n \in Z$ such that $U \cap T^{-n}(V) \neq \varnothing$.

Lemma 1.6. *Let (X, T) be a topologically transitive system and $(V, \to)$ the directed graph defined by a partition $\{O_i : i = 1, \ldots, N\}$ of X. We have*

(a) *if $i_0 \to i_1$, $j_0 \to j_1$, $i_1 \neq j_0$, $i_0 \neq j_1$ and $(i_0, i_1) \neq (j_0, j_1)$ then either $i_1 \twoheadrightarrow j_0$ or $j_1 \twoheadrightarrow i_0$,*

(b) *if $i \to i_1$, $i \to j_1 \to \cdots \to j_n$, $i \neq i_1$, $j_1, \ldots, j_n$ and $i_1 \neq j_1, \ldots, j_n$, then either $i_1 \twoheadrightarrow i$ or $j_k \twoheadrightarrow i$ for all $1 \le k \le n$.*

Proof. (a) Since $O_{i_0} \cap T^{-1}(O_{i_1})$ and $O_{j_0} \cap T^{-1}(O_{j_1})$ are two disjoint nonempty clopen subsets of X and (X, T) is topologically transitive, there exists an integer m such that

$$O_{i_0} \cap T^{-1}(O_{i_1}) \cap T^{-m}(O_{j_0} \cap T^{-1}(O_{j_1})) \neq \varnothing.$$

Under the given conditions, $m \neq -1, 0, 1$. If $m > 1$, we have

$$\begin{aligned}
&O_{i_1} \cap T^{-(m-1)}(O_{j_0}) = T(T^{-1}(O_{i_1}) \cap T^{-m}(O_{j_0})) \neq \varnothing \\
&\Rightarrow O_{i_1} \cap \left(\bigcup_{i=1}^{N} T^{-1}(O_i)\right) \cap \cdots \cap \left(\bigcup_{i=1}^{N} T^{-(m-2)}(O_i)\right) \cap T^{-(m-1)}(O_{j_0}) \neq \varnothing \\
&\Rightarrow O_{i_1} \cap T^{-1}(O_{i_2}) \cap \cdots \cap T^{-(m-2)}(O_{i_{m-1}}) \cap T^{-(m-1)}(O_{j_0}) \neq \varnothing \\
&\hphantom{\Rightarrow} \qquad\qquad \text{for some } i_2, \ldots, i_{m-1}. \\
&\Rightarrow i_1 \to i_2 \to \cdots \to i_{m-1} \to j_0 \quad \text{for some } i_2, \ldots, i_{m-1}.
\end{aligned}$$

Similarly, if $m < -1$, we have $j_1 \twoheadrightarrow i_0$.

To prove (b), we use induction on n.

(i) If $n = 1$, then the result follows from (a).

(ii) For $n > 1$, we may assume that $j_k \twoheadrightarrow i$ for all $1 \le k \le n-1$. Applying (a) to the pair $i \to i_1$ and $j_{n-1} \to j_n$, we have either $j_n \twoheadrightarrow i$ or $i_1 \twoheadrightarrow j_{n-1}$. But $i_1 \twoheadrightarrow j_{n-1}$, together with $j_{n-1} \twoheadrightarrow i$, would imply $i_1 \twoheadrightarrow i$. $\square$

Proposition 1.7. *Let (X, T) be a topologically transitive system and $P = \{O_i : i = 1, \dots, N\}$ a partition of X. If $f \in C(X, Z)$ satisfies $\sum_s f = 0$ for every cycle s of P, then for every i, j and any two paths s_1, s_2 from i to j, we have $\sum_{s_1} f = \sum_{s_2} f$.*

Proof. Let $f = \sum_{i=1}^N a_i \chi_{O_i}$, $s_1 = (i_0, i_1, \dots, i_m)$ and $s_2 = (j_0, j_1, \dots, j_n)$. We have $i_0 = j_0 = i$, $i_m = j_n = j$. If $i = j$, then $s_3 = (i_0, \dots, i_{m-1})$ and $s_4 = (j_0, \dots, j_{n-1})$ are cycles. This gives

$$\sum_{s_1} f = \sum_{s_3} f + a_j = 0 + a_j = \sum_{s_4} f + a_j = \sum_{s_2} f.$$

So, we may assume $i \neq j$. We are going to prove the result by induction on $\min(m, n)$.

(1) For $\min(m, n) = 1$, we may assume $m = 1$ and $n > 1$. If $j = j_k$ for some $1 \leq k < n$, then $s = (j_{k+1}, \dots, j_n)$ is a cycle and $s_3 = (j_0, \dots, j_k)$ is a path from i to j such that $\sum_{s_3} f = \sum_{s_3} f + \sum_s f = \sum_{s_2} f$. Thus we may assume that $j \neq j_k$ for all $1 \leq k < n$. Similarly, we may assume $i \neq j_k$ for $1 \leq k < n$. From Lemma 1.6(b), we have $j \twoheadrightarrow i$. Hence $\sum_{s_1} f = \sum_{s_2} f$ by Lemma 1.4.

(2) Suppose $m = \min(m, n) > 1$. Again, we may assume $i, j \neq i_h$, j_k for $1 \leq h < m$, $1 \leq k < n$. If $i_1 = j_k$ for some $1 \leq k < n$, then, by induction assumption, we have

$$a_{i_0} + a_{i_1} = \sum_{t=0}^{k} a_{j_t} \quad \text{and} \quad \sum_{h=1}^{m} a_{i_h} = \sum_{t=k}^{n} a_{j_t} \Rightarrow \sum_{s_1} f = \sum_{s_2} f.$$

If $i_1 \neq j_k$ for all $1 \leq k < m$, then, by Lemma 1.6(b), we have either (a) $i_1 \twoheadrightarrow i$ or (b) $j \twoheadrightarrow i$.

(a) If $i_1 \twoheadrightarrow i$, let $s = (k_1, \dots, k_r)$ be a path from i_1 to i. Since $i \to i_1$, s is a cycle and $\sum_s f = 0$. Consider the paths $s_3 = (i_1, i_2, \dots, i_m)$ and $s_4 = (k_1, \dots, k_r, j_1, \dots, j_n)$. Again, by induction assumption, we have

$$\sum_{s_1} f = a_i + \sum_{s_3} f = a_i + \sum_{s_4} f = \sum_{s} f + \sum_{s_2} f = \sum_{s_2} f.$$

(b) If $j \twoheadrightarrow i$, the result follows from Lemma 1.4.

Proposition 1.8. *Given a dynamical system (X, T), let $(V, \to)$ be the directed graph defined by a partition $\{O_i : i = 1, \dots, n\}$ of X and $a = (a_1, \dots, a_n)$, $a_i \in Z$. Then we have*

(1) *$\sum_s a \geq 0$ for every cycle s if and only if there exist integers t_i, $1 \leq i \leq n$, such that $a_i + t_i - t_j \geq 0$ for all i, j with $i \to j$. If, in addition, (X, T) is topologically transitive, then we have*

(2) *$\sum_s a = 0$ for every cycle s if and only if there exist integers t_i, $1 \leq i \leq n$, such that $a_i + t_i - t_j = 0$ for all i, j with $i \to j$.*

Proof. To prove (1), suppose there exist $t_i \in Z$, $1 \le i \le n$, such that $a_i + t_i - t_j \ge 0$ whenever $i \to j$. Then given any cycle $s = (i_1, \dots, i_m)$, we have

$$i_1 \to i_2 \to \cdots \to i_m \to i_1 \Rightarrow \begin{cases} a_{i_1} + t_{i_1} - t_{i_2} & \ge 0 \\ \vdots & \vdots \\ a_{i_j} + t_{i_j} - t_{i_{j+1}} & \ge 0 \\ \vdots & \vdots \\ a_{i_m} + t_{i_m} - t_{i_1} & \ge 0 \end{cases}$$

$$\Rightarrow \sum_s a = \sum_{j=1}^{m} a_{i_j} \ge 0.$$

Conversely, suppose $\sum_s a \ge 0$ for every cycle s. For each i, $1 \le i \le n$, define

$$t_i = \min_{m \ge 1} \left\{ \sum_{k=1}^{m} a_{i_k} : (i_1, \dots, i_m, i) \text{ is a path} \right\}.$$

Claim. t_i is well defined.

Clearly, for every $i \in V$, there is a path $(i_1, \dots, i_m, i)$ with $m \ge 1$. Suppose $(i_1, \dots, i_m, i)$ is a path with $m > n + 2$. Then there exist r, t, $1 < r < t < m$, such that $i_r = i_t$. Thus, $(i_1, \dots, i_{r-1}, i_t, \dots, i_m, i)$ is also a path and $s = (i_r, \dots, i_{t-1})$ is a cycle. So we have

$$\sum_{k=1}^{m} a_{i_k} = \sum_{k=1}^{r-1} a_{i_k} + \sum_s a + \sum_{k=t}^{m} a_{i_k} \ge \left(\sum_{k=1}^{r-1} a_{i_k} + \sum_{k=t}^{m} a_{i_k} \right).$$

Hence, in the definition of t_i, we only need to take the minimum over $1 \le m \le n + 2$. Therefore, every t_i is well defined.

For every i, j with $i \to j$, choose a path $(i_1, \dots, i_m, i)$ with $\sum_{k=1}^{m} a_{i_k} = t_i$. Then, $(i_1, \dots, i_m, i, j)$ is a path and we have

$$a_i + t_i = a_i + \sum_{k=1}^{m} a_{i_k} \ge t_j \Rightarrow a_i + t_i - t_j \ge 0.$$

To prove (2), suppose there exist t_i, $1 \le i \le n$ such that $a_i + t_i - t_j = 0$ whenever $i \to j$. Then, replacing "$\ge$" by "$=$" in the first part of the proof of (1), we have $\sum_s a = 0$ for every cycle s.

Conversely, suppose $\sum_s a = 0$ for every s.

Claim.

(a) There exists k in V such that $k \twoheadrightarrow k$

(b) For every $1 \le i \le n$, either $i \twoheadrightarrow k$ or $k \twoheadrightarrow i$.

Proof of (a). $X = \bigcap_{r=0}^{n} T^{-r}(\bigcup_{i=1}^{n} O_i) = \bigcup_{1 \le i_r \le n} (\bigcap_{r=0}^{n} T^{-r}(O_{i_r}))$. Therefore there exist $i_0, i_1, \dots, i_n$ such that

$$\bigcap_{r=0}^{n} T^{-r}(O_{i_r}) \ne \varnothing \Rightarrow i_0 \to i_1 \to \cdots \to i_n.$$

Since $1 \leq i_r \leq n$, there exist r, t, $0 \leq r < t \leq n$, such that $i_r = i_t$. Choosing $k = i_r$, we have $k \twoheadrightarrow k$.

Proof of (b). From (a), we may assume $i \neq k$. So O_i and O_k are two disjoint nonempty sets. Since (X, T) is topologically transitive, there exists $m \in Z$, $m \neq 0$ such that $O_i \cap T^{-m}(O_k) \neq \varnothing$. Hence, as in the proof of Lemma 1.6(a), we have $i \twoheadrightarrow k$ if $m > 0$ and $k \twoheadrightarrow i$ if $m < 0$.

This establishes the claim. To finish the proof of (2), we define, for each $1 \leq i \leq n$,

$$P_i = \{s : s \text{ is a path from } k \text{ to } i\},$$
$$Q_i = \{s : s \text{ is a path from } i \text{ to } k\} \text{ and}$$
$$t_i = \begin{cases} \sum_s a - a_i & \text{if there exists } s \in P_i, \\ a_k - \sum_s a & \text{if there exists } s \in Q_i. \end{cases}$$

By the choice of k, for each i, either P_i or Q_i is nonempty. If $P_i \neq \varnothing$, Proposition 1.7 shows that $\sum_s a - a_i$ does not depend on the choice of s in P_i. Similarly, $a_k - \sum_s a$ does not depend on the choice of s in Q_i. If $s_1 = (i_1, \ldots, i_m) \in P_i$ and $s_2 = (j_1, \ldots, j_r) \in Q_i$, then $s = (i_1, \ldots, i_{m-1}, j_1, \ldots, j_{r-1})$ is a cycle and $i_1 = j_r = k$, $i_m = j_1 = i$. So we have

$$0 = \sum_s a = \sum_{s_1} a - a_{i_m} + \sum_{s_2} a - a_{j_r}$$
$$\Rightarrow \sum_{s_1} a - a_i = a_k - \sum_{s_2} a.$$

Thus t_i is well defined.

Given any i, j with $i \to j$. Consider the nonempty clopen sets O_k and $O_i \cap T^{-1}(O_j)$. Let $m \in Z$ such that $O_k \cap T^{-m}(O_i \cap T^{-1}(O_j)) \neq \varnothing$. If $m \geq 0$, we have a path $s_1 = (k, \ldots, i, j) \in P_j$. Hence, $s_2 = (k, \ldots, i) \in P_i$. So we have

$$a_i + t_i = \sum_{s_2} a = \sum_{s_1} a - a_j = t_j, \qquad a_i + t_i - t_j = 0.$$

If $m < 0$, then we have a path $s_1 = (i, j, \ldots, k) \in Q_i$ with $s_2 = (j, \ldots, k) \in Q_j$. Thus

$$a_i + t_i = a_i + \left(a_k - \sum_{s_1} a\right) = a_k - \sum_{s_2} a = t_j, \qquad a_i + t_i - t_j = 0. \qquad \square$$

Combining the results in Proposition 1.2 and Proposition 1.8, we have

Proposition 1.9. *Let (X, T) be topologically transitive. Then for $f \in C(X, T)$, we have*

(1) $[f] \in G^+$ *if and only if there exists a partition P such that $\sum_s f \geq 0$ for every cycle s of P.*

(2) $[f] = 0$ *in G if and only if there exists a partition P such that $\sum_s f = 0$ for every cycle s of P.*

Given two partitions $P = \{O_i\}$ and $P' = \{Q_j\}$, we say that P' is a refinement of P $(P < P')$ if every Q_j lies in some O_i, or equivalently, every O_i is a union of Q_j.

Proposition 1.10. *Let $f \in C(X, Z)$ and $P < P'$. If $\sum_s f \geq 0$ for every cycle s in P then $\sum_{s'} f \geq 0$ for every cycle s' in P'.*

Proof. Suppose $P = \{O_i : 1 \leq i \leq n\}$. Then for each i, there exist $O(i, 1), \ldots, O(i, k_i) \in P'$ such that $O_i = \bigcup_{j=1}^{k_i} O(i, j)$ and $P' = \{O(i, j) : 1 \leq i \leq n, 1 \leq j \leq k_i\}$. So, if $f = \sum_{i=1}^n a_i \chi_{O_i}$, then, putting $a(i, j) = a_i$ for $1 \leq i \leq n$ and $1 \leq j \leq k_i$, we have

$$f = \sum_{i=1}^{n} \sum_{j=1}^{k_i} a(i, j) \chi_{O(i,j)}.$$

Suppose $s' = ((i_1, j_1), \ldots, (i_m, j_m))$ is a cycle of P'. Then, from $O(i, j) \subseteq O_i$, we get a cycle $s = (i_1, \ldots, i_m)$ of P and

$$\sum_{s'} f = \sum_{r=1}^{m} a(i_r, j_r) = \sum_{r=1}^{m} a_{i_r} = \sum_{s} f \geq 0.$$

Now, we are ready to prove the main result in this section.

Theorem 1.11. *If (X, T) is topologically transitive, then (G, G^+) is an (unperforated) ordered group* [10], *i.e. the following are satisfied*

(a) $G^+ + G^+ \subseteq G^+$,
(b) $G^+ - G^+ = G$,
(c) $G^+ \cap (-G^+) = \{0\}$,
(d) *If $g \in G$ satisfies $ng \in G^+$ for some positive integer n, then $g \in G^+$.*

Proof. (a) and (b) follow directly from the definitions. For (c), let $[f] \in G^+ \cap (-G^+)$. Then by Proposition 1.9 (1), there exist partitions P_1 and P_2 such that

$$\sum_s f \geq 0 \quad \text{for every cycle } s \text{ of } P_1$$

and

$$\sum_s (-f) \geq 0 \quad \text{for every cycle } s \text{ of } P_2.$$

Applying Proposition 1.10 to the common refinement

$$P_1 \vee P_2 = \{O_1 \cap O_2 : O_i \in P_i, O_1 \cap O_2 \neq \varnothing\}$$

of P_1 and P_2, we have

$$\begin{aligned} \sum_s f \geq 0 \text{ and } \sum_s (-f) \geq 0 \quad & \text{for every cycle } s \text{ of } P_1 \vee P_2, \\ \Rightarrow \sum_s f = 0 \quad & \text{for every cycle } s \text{ of } P_1 \vee P_2, \\ \Rightarrow [f] = 0 \quad & \text{in } G \quad \text{(by Proposition 1.9(2))}. \end{aligned}$$

For (d), let $[f] \in G$ with $n[f] = [nf] \in G^+$ for some positive integer n. So, there exists a partition P such that

$$\begin{aligned}&\Rightarrow \sum_s (nf) \geq 0 \quad \text{for every cycle } s \text{ of } P\\ &\Rightarrow \sum_s f \geq 0 \qquad \text{for every cycle } s \text{ of } P\\ &\Rightarrow [f] \in G^+. \qquad \square\end{aligned}$$

Remark 1.12. Suppose Γ is the directed graph defined [4] by

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

If we choose $a = (0,1,0)$ and two paths $s_1 = (1,2,3)$, $s_2 = (1,3)$, then we have $\sum_s a = 0$ for every cycle s but $\sum_{s_1} a \neq \sum_{s_2} a$ and there exists no $t = (t_1, t_2, t_3)$ such that $a_i + t_i - t_j = 0$ for every $i \to j$. Furthermore, if we let (X, T) be the Markov chain defined [9] by A, then it can be shown that $G^+ \cap (-G^+) \neq \{0\}$. Thus (G, G^+) is not an ordered group.

2. The state space $S(G)$

Suppose (G, G^+) is an ordered group. For $g, h \in G$, we write $g \geq h$ if $g - h \in G^+$. An element $u \in G^+$ is called an *ordered unit* if for every $g \in G^+$, there exists $n > 0$ such that $nu \geq g$. The triple (G, G^+, u) is called a *unital ordered group.* The *state space* $S_u(G)$ of (G, G^+, u) is the set of all homomorphisms p from G to the real numbers R such that $p(g) \geq 0$ for all $g \in G^+$ and $p(u) = 1$. When $G = G(X, T)$, we always choose the ordered unit $u = [1]$, the class of the constant function 1. In this case, we just write $S(G)$ for $S_{[1]}(G)$. $S(G)$ is a compact convex subset of R^G, the space of all functions $f: G \to R$ with the product topology.

Given $p \in S(G)$, define $\mu_p(O) = p([\chi_O])$ for all clopen subsets O of X. μ_p extends to a Borel measure on X with $\mu_p(X) = 1$. Furthermore, for each clopen O, we have

$$\mu_p(T^{-1}(O)) = p([\chi_{T^{-1}(O)}]) = p([\chi_O \circ T]) = p([\chi_O]) = \mu_p(O).$$

Therefore, $\mu_p \in M(X, T)$, the set of T-invariant probability measures on X. Conversely, given $\mu \in M(X, T)$, we can define p_μ in $S(G)$ by

$$p_\mu\left(\left[\sum_i a_i \chi_{O_i}\right]\right) = \sum a_i \mu(O_i).$$

In summary, we have

Proposition 2.1. *$S(G) = \{p_\mu : \mu \in M(X, T)\}$. Furthermore, the map $p \to \mu_p$ gives a one-to-one correspondence between $S(G)$ and $M(X, T)$.*

Proof. For $p \in S(G)$ and $\mu \in M(X, T)$, one checks easily that $p_{(\mu_p)} = p$ and $\mu_{(p_\mu)} = \mu$. This establishes the second statement and completes the proof. $\square$

$M(X, T)$ is a *Choquet simplex* [11]. Since $\mu_{(tp+(1-t)q)} = t\mu_p + (1-t)\mu_q$ for $p, q \in S(G)$ and $0 \le t \le 1$, extreme points in $S(G)$ correspond to those of $M(X, T)$. Sometimes we can determine $S(G)$ directly through this correspondence.

Proposition 2.2. *If $M(X, T)$ contains more than one point and the set of ergodic measures is dense in $M(X, T)$, then $S(G)$ is isomorphic to the Poulsen simplex* [2].

Proof. Since $\mu \in M(X, T)$ is an extreme point if and only if μ is ergodic (see Theorem 6.10 of [25]), the result follows from the fact that the Poulsen simplex is the only simplex (other than a single point) for which the set of extreme points is dense [2].

If $M(X, T)$ contains only one point, then the system (X, T) is said to be *uniquely ergodic* (see [25]). On the other hand, the set of ergodic measures is dense in $M(X, T)$ if (X, T) satisfies the Specification Property in [9], e.g. if (X, T) is a Markov chain defined by a matrix A such that for some n, all entries of A^n are positive. Hence, many systems have the same $S(G)$. However, we are going to show that many of these systems can still be distinguished by the order in G. This will come as a corollary to the following

Theorem 2.3. *Let $p \in ES(G)$, the set of extreme points of $S(G)$. Then the following are equivalent*:

(1) $\inf(p) = \inf\{p(g) \colon g \in G^+, p(g) > 0\} > 0$.
(2) $n = 1/\inf(p)$ *is an integer and* μ_p *is a* T*-invariant atomic measure on an orbit of period* n.
(3) $\min(p) = \min\{p(g) \colon g \in G,\ p(g) > 0\} = 1/n$.

Proof. (1) $\Rightarrow$ (2)

Suppose $\inf(p) = r > 0$. Since every $g \in G^+$ has the form $g = [\sum_{i=1}^m a_i \chi_{O_i}]$, where a_i are nonnegative integers and $p(g) = \sum_{i=1}^m a_i \mu_p(O_i)$, we have

$$\inf\{\mu_p(O) \colon O \text{ clopen in } X,\ \mu_p(O) > 0\} = r > 0.$$

Therefore, there exists a nonempty clopen set O with $r \le \mu_p(O) = s < 2r$. Let $\{Q_i \colon 1 \le i \le m\}$ be a partition of O into clopen sets of diameter $\le 2^{-1}$. Then $s = \sum_{i=1}^m \mu_p(Q_i) < 2r$. But, for each i, $\mu_p(Q_i)$ is either 0 or $\ge r$. So, there exists a unique i such that $\mu_p(Q_i) = s$. Let $O_1 = Q_i$. Similarly, we can find a clopen set $O_2 \subseteq O_1$ such that $\mu_p(O_2) = s$ and diameter $O_2 \le 2^{-2}$. Repeating this process, we get a sequence of clopen sets $O \supseteq O_1 \supseteq O_2 \supseteq \cdots$ such that $\mu(O_n) = s$ for all n and diameter $O_n \le 2^{-n}$. Thus $\bigcap_{n=1}^\infty O_n = \{x_0\}$ for some $x_0 \in X$ and $\mu_p(\{x_0\}) = s$. Therefore, $\mu_p(\{T^k(x_0)\}) = s$ for all k in Z. Hence the set $Y = \{T^k(x_0) \colon k \in Z\}$ is finite. Suppose Y has n elements. We have $\mu_p(Y) = ns > 0$. Since Y is T-invariant and μ_p is ergodic, we have

$1 = \mu_p(Y) = ns \Rightarrow s = 1/n$. So, μ_p is a T-invariant atomic measure on the orbit of x_0 and $r = s$.

Clearly, (2) implies (3).

For (3) $\Rightarrow$ (1), suppose $\min(p) = 1/n$. Then we have $\inf(p) \geq \min(p) = 1/n > 0$. $\square$

Let G and H be ordered groups with ordered units u and v respectively. A *unital order homomorphism* between G and H is a group homomorphism $f: G \to H$ such that $f(G^+) \subseteq H^+$ and $f(u) = v$. If, in particular, f is bijective and $f(G^+) = H^+$, then we call f a *unital order isomorphism* and $(G, G^+, u), (H, H^+, v)$ are said to be (*order*) *isomorphic*. A property of unital ordered groups is said to be an *invariant* if isomorphic groups have the same property.

Proposition 2.4. *For each* $n \geq 1$, *define*

$$\mathrm{Per}(n) = \textit{cardinality of}\, \{x \colon T^n(x) = x\}.$$

Then the sequence $\{\mathrm{Per}(n) \colon n \geq 1\}$ *is an invariant of* G.

Proof. Suppose $x \in X$ satisfies $T^n(x) = x$. Let $k = \min\{m > 0 \colon T^m(x) = x\}$. Then k divides n (denoted by $k|n$) and there is a unique ergodic T-invariant measure μ_x on the orbit of x with $\mu_x(\{x\}) = 1/k$. Hence $p_{\mu_x} \in ES(G)$ and $\min(p_{\mu_x}) = 1/k$. Also, it is easy to see that for $x, y \in X$ with $\min(p_{\mu_x}) = \min(p_{\mu_y}) = 1/k$, we have $p_{\mu_x} = p_{\mu_y}$ if and only if $y = T^i(x)$ for some i, $0 \leq i < k$.

Conversely, if $k|n$ and $p \in ES(G)$ satisfies $\min(p) = 1/k$, then by Theorem 2.3, there exists $x \in X$ such that $T^k(x) = x$ and $p = p_{\mu_x}$. So, by defining

$$N_k = \text{cardinality of } \{p \in ES(G) \colon \min(p) = 1/k\}$$

for each $k \geq 1$, we have

$$\mathrm{Per}(n) = \sum_{k|n} kN_k.$$

Thus, $\{\mathrm{Per}(n) \colon n \geq 1\}$ is an invariant of G.

Remark 2.5. Let (X, T) be a subshift of a finite shift [9]. Then $\mathrm{Per}(n)$ is finite for each $n \geq 1$ and the zeta function $\zeta(z)$ of the system (X, T) is defined as [5]

$$\zeta(z) = \exp\left(\sum_{n=1}^{\infty} \frac{\mathrm{Per}(n)}{n} z^n\right).$$

Thus, from Proposition 2.4, $\zeta(z)$ is an invariant of $G(X, T)$.

Remark 2.6. Every $n \times n$ matrix A with nonnegative integral entries defines a Markov chain (X_A, T_A). (X_A, T_A) is topologically transitive if A is irreducible [9]. Suppose A is an irreducible matrix such that $G(X_A \cdot T_A)$ is isomorphic to $G(X_N, T_N)$ for a positive integer N. Then Remark 2.5 shows that (X_A, T_A) and (X_N, T_N) have the same zeta function. By a result of Williams [26, §8], the matrices A and N are shift equivalent (see Effros [10] for a discussion of the relation to a conjugacy problem, the Williams's problem).

Remark 2.7. Given an irreducible matrix A, let B be the transpose of A. It is easy to show that $G(X_A, T_A)$ and $G(X_B, T_B)$ are isomorphic. But, as Cuntz and Krieger have shown [8], A and B are usually not even shift equivalent. Thus, systems with isomorphic ordered groups need not be conjugate to each other. However, a weaker form of conjugacy is possible. A notion of almost topological conjugacy has been introduced by Adler and Marcus in [1], where they prove that two irreducible Markov chains are almost topologically conjugate if and only if they have the same period and entropy. Since the period of a Markov chain is equal to the greatest common divisor of all $n \geq 1$ such that $\text{Per}(n) > 0$ and the entropy is given by [9]

$$\lim_{n\to\infty}\left(\frac{\log \text{Per}(n)}{n}\right),$$

it follows that if $G(X_A, T_A)$ is isomorphic to $G(X_B, T_B)$, then (X_A, T_A) and (X_B, T_B) are almost topologically conjugate to each other.

Remark 2.8. Given two topologically transitive dynamical systems (X, T) and (Y, S), (Y, S) is said to be a *factor* of (X, T) if there exists a continuous surjection $\phi\colon X \to Y$ such that $\phi\circ T = S\circ\phi$. Then $\phi^*([f]) = [f\circ\phi]$ defines a unital order homomorphism from $G(Y, S)$ to $G(X, T)$. If (Y, S) is a Markov chain, then, using the properties of Markov partitions [15], one can show that ϕ^* is one to one and

$$\phi^*(G^+(Y, S)) = G^+(X, T)\cap\phi^*(G(Y, S)).$$

Hence, $G(Y, S)$ is isomorphic to an ordered subgroup of $G(X, T)$.

3. Direct limit of unital ordered groups

In this section, we are going to show that G can be computed from a sequence of finite rank unital ordered groups.

Definition 3.1 [10]. Suppose $\{(G_n, G_n^+, u_n)\}_{n=1}^\infty$ is a sequence of unital ordered groups and $\Phi_n\colon G_n \to G_{n+1}$ is a unital order homomorphism for each n. Let $\bigsqcup_{n=1}^\infty G_n = \{(g, n)\colon g\in G_n\}$ be the disjoint union of $\{G_n\}$. For $n > m$, define $\Phi_{mn}\colon G_m \to G_n$ by $\Phi_{mn} = \Phi_{n-1}\circ\Phi_{n-2}\circ\cdots\circ\Phi_m$. Then define an equivalence relation R on $\bigsqcup_{n=1}^\infty G_n$ by $(g, n)\,R\,(h, m)$ if there exists $k > n, m$ such that $\Phi_{nk}(g) = \Phi_{mk}(h)$. One can easily verify that R is indeed an equivalence relation. Let $G = \bigsqcup_{n=1}^\infty G_n/R$. Denote the equivalence class of (g, n) by $[g, n]$. A group structure can be defined on G by $[g, n] + [h, m] = [\Phi_{nk}(g) + \Phi_{mk}(h), k]$ where k is any integer $> n, m$. Taking $G^+ = \{[g, n] : g\in G_n^+\}$, we have an ordering on G. Since each Φ_n is unital, $u = [u_n, n]$ is independent of n and gives an order unit of G. The unital ordered group (G, G^+, u) is called the *direct limit* of $\{(G_n, \Phi_n)\}_{n=1}^\infty$ and denoted by $\varinjlim(G_n, \Phi_n)$.

Definition 3.2. Suppose Γ is a directed graph on a finite set V. Let $C_\Gamma = \{f: V \to Z\}$ and $B_\Gamma = \{f \in C_\Gamma: \sum_s f = 0$, for every cycle s of $\Gamma\}$. Then C_Γ is a group under the usual addition and B_Γ is a subgroup of C_Γ. Define $G_\Gamma = C_\Gamma/B_\Gamma$ and

$$G_\Gamma^+ = \left\{[f]: \sum_s f \geq 0 \text{ for every cycle } s \text{ of } \Gamma\right\}.$$

Then (G_Γ, G_Γ^+) is an ordered group with order unit $[1]$, the class of the constant function 1.

Let (X, T) be a zero-dimensional dynamical system. Since X is separable, there exists a sequence of finite partitions $P_1 < P_2 < \cdots < P_n \cdots$ such that $\bigcup_{n=1}^\infty P_n$ is a basis for the topology of X. Suppose $P_n = \{O_v: v \in V_n\}$. Let Γ_n be the directed graph on V_n associated with the partition P_n (see Definition 1.3) and $G_n = G_{\Gamma_n}$: Then, since $P_n < P_{n+1}$, there exists a unique map ϕ_n of V_{n+1} onto V_n such that for every v' of $V_n, O_{v'} = \bigcup\{O_v: v \in \phi_n^{-1}(v')\}$. This gives a map $\Phi_n: G_n \to G_{n+1}$ by $\Phi_n([f]) = ([f \circ \phi_n])$. One checks that Φ_n is a well-defined unital order homomorphism for each n. Hence we can define the direct limit of $\{(G_n, \Phi_n)\}_{n=1}^\infty$. The main result in this section is

Theorem 3.3. *Suppose (X, T) is topologically transitive. Let $P_1 < P_2 < \cdots$ be a sequence of partitions of X such that $\bigcup_{n=1}^\infty P_n$ is a basis for the topology of X. Then $G(X, T)$ is order isomorphic to $\varinjlim(G_n, \Phi_n)$.*

Proof. Since $\bigcup_{n=1}^\infty P_n$ is a basis for the topology of X, every partition P has a refinement P_n for sufficiently large n. Thus, given $f \in C(X, Z)$, we can find n such that $f = \sum_{v \in V_n} a_v \chi_{O_v}$, where $P_n = \{O_v: v \in V_n\}$. Define $f_n \in C_{\Gamma_n}$ by $f_n(v) = a_v$ for $v \in V_n$. We have $f_{n+1} = f_n \circ \phi_n$. Let $H = \varinjlim(G_n, \Phi_n)$. We are going to prove that the map $\psi: G(X, T) \to H$ defined by $\psi([f]) = [f_n] \in H$ is a unital order isomorphism.

Since $f_{n+1} = f_n \circ \phi_n$, ψ is well defined. Clearly, ψ is onto and $\psi([1]) = [1]$. Furthermore, we have

$$\begin{aligned}
[f] \in G^+ &\Leftrightarrow \text{for some } n, \ \sum_s f \geq 0 \quad \text{for every cycle } s \text{ of } \Gamma_n \\
&\Leftrightarrow \text{for some } n, \ \sum_s f_n \geq 0 \quad \text{for every cycle } s \text{ of } P_n \\
&\Leftrightarrow [f_n] \in G_n^+ \quad \text{for some } n \\
&\Leftrightarrow \psi([f]) \geq 0 \quad \text{in } H.
\end{aligned}$$

This gives $G^+ = \psi^{-1}(H^+)$. Thus, ψ is one to one and hence is bijective. Therefore ψ is a unital order isomorphism. □

Example 3.4. Let $X = \{-\infty, \infty\} \cup Z$ be the 2-point compactification of the integers Z. For each N, define a partition

$$P_N = \{O(N, -\infty), O(N, \infty), O(N, k): -N < k < N\}$$

where

$$O(N, -\infty) = \{k : k \leq -N\} \cup \{-\infty\},$$
$$O(N, \infty) = \{k : k \geq N\} \cup \{\infty\},$$
$$O(N, k) = \{k\} \quad \text{for } -N < k < N.$$

Then, by definition, $\bigcup_N P_N$ is a clopen basis for X and X is zero dimensional. Define $T: X \to X$ by

$$T(\infty) = \infty, \quad T(-\infty) = -\infty, \quad T(k) = k+1, \qquad k \in Z.$$

Therefore $\{T^k(0): k \in Z\}$ is dense in X and (X, T) is topologically transitive. For each N, the directed graph Γ_N defined by P_N is

$$\begin{tikzcd} (-\infty) \arrow[loop left] \arrow[r] & -N+1 \arrow[r] & \cdots \arrow[r] & N-1 \arrow[r] & \infty \arrow[loop right] \end{tikzcd}$$

This gives $G_N = Z^2$, $G_N^+ = Z_+^2 = \{(a, b): a, b \geq 0\}$ and $[1] = (1, 1)$. The map $\Phi_N: G_N \to G_{N+1}$ is the identity. Hence $[f] \to (f(\infty), f(-\infty))$ gives an isomorphism of $G(X, T)$ and $(Z^2, Z_+^2, (1, 1))$.

Example 3.5. Let $Y = \{\infty\} \cup Z$ be the 1-point compactification of the integers. Define $S: Y \to Y$ by

$$S(\infty) = \infty \quad \text{and} \quad S(n) = n+1 \qquad \text{for } n \in Z.$$

For each N, we take the partition

$$Q_N = \{Q(N, \infty), Q(N, k): -N < k < N\}$$

where

$$Q(N, k) = \{k\} \quad \text{and} \quad Q(N, \infty) = Y \setminus \bigcup_k Q(N, k).$$

Again, $\bigcup_N Q_N$ is a clopen basis of Y and (Y, S) is topologically transitive. Q_N determines a directed graph:

$$\begin{tikzcd} & \infty \arrow[loop above] \arrow[dl] & & \\ -N+1 \arrow[r] & -N+2 \arrow[r] & \cdots \arrow[r] & (N-1). \arrow[ull] \end{tikzcd}$$

For

$$f = a_\infty \chi_{Q(N, \infty)} + \sum_{k=-N+1}^{N-1} a_k \chi_{Q(N, k)},$$

we can identify $[f_N]$ with $(a_\infty, a_\infty + \sum_{k=-N+1}^{N-1} a_k) \in Z^2$. This gives $G_N = Z^2$, $G_N^+ = Z_+^2$, $[1] = (1, 2N)$ and the map $\Phi_N\colon G_N \to G_{N+1}$ is given by

$$\Phi_N(a, b) = (a, b + 2a).$$

Thus $(G, G^+, [1]) = \varinjlim(G_N, \Phi_N) = (Z^2, H, (1, 0))$ where $H = \{(a, b) \in Z^2\colon a > 0 \text{ or } a = 0,\ b \geq 0\}$. Here, $(a, b) \in G_N$ is identified with $(a, b - 2Na) \in G$. So, if $f = a_\infty \chi_{Q(N,\infty)} + \sum_{k=-N+1}^{N-1} a_k \chi_{Q(N,k)}$, then $[f] = (a_\infty, \sum_{k=-N+1}^{N-1}(a_k - a_\infty)) \in G$.

Remark 3.6. Define a continuous surjection $\phi\colon X \to Y$ by

$$\phi(\infty) = \phi(-\infty) = \infty \quad \text{and} \quad \phi(k) = k \qquad \text{for } k \in Z.$$

Then $\phi \circ T = S \circ \phi$. Hence, (Y, S) is a factor of (X, T). But it can be shown that neither one of $G(X, T)$ and $G(Y, T)$ is isomorphic to an ordered subgroup of the other. This shows that the conclusion in Remark 2.8 may not hold if (Y, S) is not a Markov chain. Incidentally, it can be shown that the two systems in the above examples are subshifts of the 2-shift and that (X, T) is the Markov chain defined by $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$.

Remark 3.7. An ordered group G is said to have the *Riesz interpolation property* if given $a_1, a_2, b_1, b_2 \in G$ with $a_i \leq b_j$ $(i, j = 1, 2)$, there exists $c \in G$ such that $a_i \leq c \leq b_j$ $(i, j = 1, 2)$. The theorem [12] of Effros, Handelman and Shen says that a countable ordered group is a dimension group if and only if it has the Riesz interpolation property. Thus, given a zero-dimensional dynamical system (X, T) we want to determine when $G(X, T)$ will have the Riesz interpolation property. By Theorem 3.3, we can assume that G is given by $\varinjlim(G_n, \Phi_n)$ where G_n is given by a sequence of partitions. Thus if every G_n has the Riesz interpolation property, then G will be a dimension group.

Example 3.8. *The adding machine.* Let $\{n_k\colon k \geq 1\}$ be a sequence of positive integers > 1. Define $X = \prod_{k=1}^\infty \{0, 1, \ldots, n_k - 1\}$ with the product topology and $T\colon X \to X$ by

$$(T(x))_k = \begin{cases} 0 & \text{if } x_i = n_i - 1 \text{ for all } i \leq k, \\ x_k + 1 & \text{if } x_i = n_i - 1 \text{ for all } i < k \text{ and } x_k < n_k - 1, \\ x_k & \text{if } x_i < n_i \text{ for some } i < k. \end{cases}$$

Then (X, T) is called the *adding machine* [24] associated to the sequence $\{n_k\colon k \geq 1\}$. The crossed product $Z \times_T C(X)$ is a simple C^*-algebra given by a direct limit $\varinjlim M_{n_k}(C(S))$, where S is the unit circle. This class of C^*-algebras has been studied by Bunce and Deddens in [6], where it is proved that the generalized natural number $\prod_{k=1}^\infty n_k$ (see Effros [11]) is a complete invariant for the C^*-algebras in this class. We are going to prove that G is the dimension group associated to $\prod_{k=1}^\infty n_k$ [10], i.e. G is the direct limit of $Z \xrightarrow{n_1} Z \xrightarrow{n_2} Z \to \cdots \to$.

For each $N \geq 1$ and $a_1, \ldots, a_N$, $0 \leq a_i < n_i$. Let $[a_1, \ldots, a_N] = \{x \in X: x_i = a_i,\ 1 \leq i \leq N\}$. Define a partition $P_N = \{[a_1, \ldots, a_N]: 0 \leq a_i < n_i$ for $1 \leq i \leq N\}$. Then the corresponding directed graph Γ_N is given by

$$\begin{array}{ccccc} [0, \ldots, 0] & \longrightarrow & [1, 0, \ldots, 0] \rightarrow \cdots & & [0, \ldots, 0, 1] \\ \uparrow & & & & \uparrow \\ [a_1 - 1, \ldots, a_N - 1] & \longleftarrow & \cdots & \longleftarrow & [1, 0, \ldots, 0, 1]. \end{array}$$

Therefore $(G_N, G_N^+) \simeq (Z, Z_+)$ is a dimension group. Since $\bigcup_{N=1}^{\infty} P_N$ is a basis of the topology on X, by Theorem 3.3, G is a dimension group. Furthermore, from

$$[a_1, \ldots, a_{N-1}] = \bigcup_{a=0}^{n_N - 1} [a_1, \ldots, a_{N-1}, a].$$

The map $G_{N-1} \to G_N$ is multiplication by n_N.

Example 3.9. *The irrational rotation algebra.* Let S be the circle by identifying the end points of the unit interval $[0, 1]$. For each irrational θ define a homeomorphism $T_\theta: S \to S$ by $T_\theta(x) = x + \theta \pmod 1$. Then the crossed product $Z \times_{T_\theta} C(S)$, usually denoted by A_θ, is called the *irrational rotation algebra* of θ. A_θ is a simple C^*-algebra which has been studied extensively (see [11, 19, 23]). Of particular interest is the image of $K_0(A_\theta)$ under the homomorphism τ_* induced by the unique trace τ on A_θ. Pimsner and Voiculescu, by imbedding A_θ into an AF algebra [19], prove that $\tau_*(K_0(A_\theta)) \subseteq Z + \theta Z$ as a subset of the real line. In [23], Rieffel constructs explicitly projections in A_θ and completes the proof that $\tau_*(K_0(A_\theta)) = Z + \theta Z$. To give a simple proof of Pimsner and Voiculescu's result, Cuntz [7] introduces the following C^*-algebra:

Let $\mathbf{D}$ be the C^*-algebra of functions on S generated by $\{\chi_{[0,\theta)} \circ T_\theta^n : n \in Z\}$. Then $A_\theta = Z \times_{T_\theta} C(S) \subseteq Z \times_{T_\theta} \mathbf{D}$. Cuntz proves that $\tau'_*(K_0(Z \times_{T_\theta} \mathbf{D})) = Z + \theta Z$, where τ' is the extension of τ to $Z \times_{T_\theta} \mathbf{D}$. Since $\mathbf{D}$ is generated by projections, there is a zero-dimensional space X such that $\mathbf{D} \simeq C(X)$ and T_θ induces a homeomorphism T, on X. We are going to prove that $G(X, T)$ is the dimension group $Z + \theta Z$.

Given $n \in Z$, there exists a unique $\overline{n\theta} \in [0, 1)$ such that $n\theta - \overline{n\theta} \in Z$. Let $R = \{\overline{n\theta}: n \in Z\}$. Given $r_1, r_2 \in R$ with $r_1 < r_2$ let $[r_2, r_1) = S \backslash [r_1, r_2)$. Then $C(X)$ is equal to the closed linear span of $\{\chi_{[r_1, r_2)}: r_1, r_2 \in R\}$. Each $\chi_{[r_1, r_2)}$ is a projection in $C(X)$. Therefore, the set $I_{r_1, r_2} = \{x \in X: \chi_{[r_1, r_2)}(x) = 1\}$ is a clopen subset. Hence, we can regard subsets of the form $[r_1, r_2)$, $r_1, r_2 \in R$, as clopen subsets of X. Under this correspondence, a partition P of S by sets of the form $[r_1, r_2)$, $r_1, r_2 \in R$, gives a clopen partition P' of X. Furthermore, P and P' give the same directed graph. Thus, we are going to compute $G(X, T)$ via partitions of S by sets of the form $[r_1, r_2)$, $r_1, r_2 \in R$.

To construct the partitions, we need some results on continued fractions (see [14]):

Let θ be given by the continued fraction $1/a_1 + 1/a_2 + \cdots$. Define

$$\begin{aligned} p_0 &= 0, \quad p_1 = 1, \quad p_n = a_n p_{n-1} + p_{n-2} \quad \text{for } n \geq 2, \\ q_0 &= 1, \quad q_1 = a_1, \quad q_n = a_n q_{n-1} + q_{n-2} \quad \text{for } n \geq 2. \end{aligned}$$

For each n, let $Q_n = \{\overline{m\theta}\colon -q_n \leq m < q_{n-1}\}$. Then Q_n gives a partition, P_n of S into $q_n + q_{n-1}$ sets of the form $[r_1, r_2)$, $r_1, r_2 \in Q_n$.

The directed graphs corresponding to these partitions are

(1)

$$\begin{tikzcd}
\left[\overline{(-q_n\theta)}, 0\right) \arrow[r] & \left[\overline{(1-q_n)\theta}, \theta\right) \arrow[r] & \cdots \arrow[r] & \left[\overline{(q_{n-1}-1-q_n)\theta}, \overline{(q_{n-1}-1)\theta}\right) \arrow[d] \\
& \left[\overline{(q_{n-1}-1)\theta}, \overline{(-\theta)}\right) \arrow[u] \arrow[r] \arrow[rr, bend left] & \cdots \arrow[r] & \left[\overline{(q_{n-1}-q_n)\theta}, \overline{(-q_n\theta)}\right)
\end{tikzcd}$$

when n is even and

(2)

$$\begin{tikzcd}
\left[0, \overline{(-q_n\theta)}\right) \arrow[r] & \left[\theta, \overline{(1-q_n)\theta}\right) \arrow[r] & \cdots \arrow[r] & \left[\overline{(q_{n-1}-1)\theta}, \overline{(q_{n-1}-1-q_n)\theta}\right) \arrow[d] \\
& \left[\overline{(-\theta)}, \overline{(q_{n-1}-1)\theta}\right) \arrow[u] \arrow[rr, bend left] & \cdots \arrow[l] & \left[\overline{(-q_n\theta)}, \overline{(q_{n-1}-q_n)\theta}\right) \arrow[l]
\end{tikzcd}$$

when n is odd. It follows that for each n, $(G_n, G_n^+) \simeq (Z^2, Z_+^2)$, therefore $G = \varinjlim G_n$ is a dimension group.

To compute G explicitly, we notice from the definition of $\{a_n\}$ (see [14]) that the partition P_{n+1} is obtained by splitting each interval in P_n of length $\overline{(-1)^{n-1} q_{n-1}\theta}$ into a_{n+1} intervals of length $\overline{(-1)^n q_n \theta}$ and 1 interval with length $\overline{(-1)^{n+1} q_{n+1}\theta}$. Then one can show that G is isomorphic to

$$Z^2 \xrightarrow{\begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix}} Z^2 \xrightarrow{\begin{bmatrix} a_2 & 1 \\ 1 & 0 \end{bmatrix}} Z^2 \to \cdots.$$

Hence, by a construction of Effros and Shen [13], $G \simeq Z + \theta Z$ as an ordered subgroup of the real line.

We conclude with some remarks on the relation between $G(X, T)$ and the K-theory of the crossed product $Z \times_T C(X)$. A detailed discussion of K-theory for a C^*-algebra can be found in Blackardar [3] and Effros [10].

Remark 3.10. Since $K_1(C(X)) = 0$ for a zero-dimensional space X, the Pimsner and Voiculescu six-term exact sequence [18] shows that $G(X, T)$ is isomorphic to $K_0(Z \times_T C(X))$ as unordered groups. Given a C^*-algebra A, $K_0(A)$ is generated by a distinguished semisubgroup [3] $V(A)$, consisting of classes of projections. It would be desirable to know if $V(Z \times_T C(X)) = G^+(X, T)$ because a positive answer would imply that the unital ordered group $G(X, T)$ is an invariant of the crossed product $Z \times_T C(X)$. One always has $G^+ \subseteq V(Z \times_T C(X))$ and $p(g) \geq 0$ for every $g \in V(Z \times_T C(X))$ and $p \in S(G)$. Thus a sufficient condition for $P(Z \times_T C(X)) = G^+(X, T)$ would be $G^+ = \{g \in G: p(g) \geq 0$ for all $p \in S(G)\}$. This condition is satisfied by all Markov chains and the systems in Examples 3.8 and 3.9 but not by (Y, S) in Example 3.5. However, one can still show that the sequence $\{\text{Per}(n): n \geq 1\}$ is always an invariant of $Z \times_T C(X)$. Hence, the results in Remarks 2.5, 2.6, and 2.7 can also be formulated in terms of $Z \times_T C(X)$.

Remark 3.11. Let (X, T) be a (not necessarily zero-dimensional) dynamical system and $\{O_i: i = 1, \ldots, n\}$ be an open cover of X. Following Definition 1.3, we can define a directed graph on $\{1, \ldots, n\}$ by letting $i \to j$ if $O_i \cap T^{-1}(O_j) \neq \varnothing$. Similarly, we can define the path and cycle of $\{O_i\}$. Given a cycle $(i(1), \ldots, i(k))$, the sequence $\{O(i(j)): 1 \leq j \leq k\}$ is known as a periodic *pseudo-orbit* of $\{O_i\}$. This notion was used by Pimsner [17] to characterize systems (X, T) for which $Z \times_T C(X)$ can be unitally embedded into an AF algebra. In a future work, we are going to investigate embeddings in the other direction for zero-dimensional systems (X, T). Specifically, we want to study AF algebras A such that $C(X) \subseteq A \subseteq Z \times_T C(X)$. Since for an AF algebra A, $(K_0(A), V(A), [1])$ is a dimension group, the situation is more interesting when $G(X, T)$ is a dimension group. For systems including those in Examples 3.5, 3.8 and 3.9, the AF A algebra can even be chosen so that $(K_0(A), V(A), [1])$ is order isomorphic to $(G, G^+, [1])$. Finally, we note that the system (X, T) in Example 3.6 is given by Pimsner as one for which $Z \times_T C(X)$ cannot be unitally embedded into any AF algebra.

Remark 3.12. After the above work has been completed, we receive two preprints [21, 22] by Putnam. He proves, among other results, that if X does not have isolated points and the system (X, T) is minimal, then (1) every nonempty closed subset Y of X gives rise to an AF algebra A_Y such that $C(X) \subseteq A_Y \subseteq Z \times_T C(X)$ and (2) when Y consists of a single point, $(K_0(A_Y), V(A_Y), [1])$ is order isomorphic to $(G, G^+, [1])$. (1) has been generalized to all zero-dimensional dynamical systems [20].

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